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A GENERAL FRAMEWORK FOR THE ANALYSIS OF CONCEPT IDENTIFICATION TASKS *

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A general framework is presented for concept identification based on hypothesis-testing theory. It is a modification of the duoprocess theory presented by Chumbley (1972). It is shown how Markov models for various complex concept identification tasks may be derived from this framework and how such models may be analyzed by making use of probability generating functions. Two experiments are described. In experiment 1 three tasks were used: two simple tasks, where the subject either only had to select the relevant dimension in order to solve the problem or only had to learn a short list of paired-associates, and a more complex task, where both processes were needed to reach the solution. The results were in general favorable to the theory. Experiment 2 was designed to test the application of the theory to the four-choice concept problem. The predictions of the theory are compared to those of the subproblem learning theory of trabasso and Bower (1964), modified to include a 'learning-on-errors' assumption. The fit of the duoprocess theory was reasonably good and superior to that of the subproblem learning theory.

Introduction

Concept identification refers to the learning of a rule which maps stimuli onto response-categories (*cf.* Hunt 1962; Millward and Wickens 1974). It is a human analogue to animal discrimination learning. Despite this close relationship between human and animal discrimination learning the theoretical developments in these two areas have in general proceeded quite independently (*e.g.* Levine 1969; Brown 1974; Lovejoy 1968; Sutherland and Mackintosh 1971). In this paper we will be con-

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cerned exclusively with human discrimination learning. The stimuli that are used in these tasks are usually supposed to be dimensionally structured, that is: every stimulus can be fully described by its values on a number of (possibly subjective) dimensions.

In most cases the experimenter is able to control the subjective dimensions by using stimuli with obvious dimensions or by telling the subjects which dimensions are to be considered. Thus, the stimulus dimensions are assumed to be separable (Garner 1976). These dimensions may be partitioned into *relevant* and *irrelevant* dimensions: relevant dimensions are involved in the categorization rule, irrelevant dimensions are not involved in this rule. That is, relevant dimensions control the correct mapping of stimuli onto categories.

In principle, there is quite a variety of possible categorization rules (see Hunt 1962; Haygood and Bourne 1965). Two of the most frequently used types are (1) one-dimensional or affirmative rules and (2) conjunctive rules. Affirmative rules are the most common. In the prototypical experiment there are a number of binary dimensions (*e.g.* color, size, shape *etc.*) and one of these controls the mapping. For example, if A and B are the two values on the relevant dimension, the rule might be as follows: all stimuli that have value A on the relevant dimension, should be put into category 1, all stimuli that have value B, into category 2, or schematically:

$A \rightarrow 1 \qquad B \rightarrow 2$

This particular type of task will be referred to as the standard task.

Although this type of concept identification task is by far the most commonly used one, it is possible to use dimensions with more than two values or to use more than two response-categories. For example, possible one-dimensional rules are:

$A \rightarrow 1$	$A \rightarrow 1$	$A \rightarrow 1$	
$B \rightarrow 2$	$B \rightarrow 2$	$B \rightarrow 1$	$A \rightarrow 1$ or 2
$C \rightarrow 3$	$C \rightarrow 2$	$C \rightarrow 2$	$B \rightarrow 3$ or 4
$D \rightarrow 4$	$D \rightarrow 2$	$D \rightarrow 2$	

In the conjunctive case there are two relevant dimensions which must be used simultaneously. Again, binary dimensions are most commonly used. Conjunctive rules usually refer to tasks of the following kind:

$A_1A_2 \rightarrow 1 \qquad B_1A_2 \rightarrow 2$

$A_1B_2 \rightarrow 2 \qquad B_1B_2 \rightarrow 2$

where A_i and B_i are the two values on dimension i. Category '1' corresponds to the response 'Yes, this is an instance of the concept'. The other category corresponds to the response 'No, this is not an instance'. Another type of conjunctive rule is the so-called 'four choice concept problem':

$A_1A_2 \rightarrow 1 \qquad B_1A_2 \rightarrow 3$

$A_1B_2 \rightarrow 2 \qquad B_1B_2 \rightarrow 4$

A variety of theories and models have been proposed to explain the behavior of subjects in solving such concept-identification problems, most of them based on the notion that subjects select and test hypotheses about the correct rule. Based on the work of Lashley (1942) and Krechevsky (1932), models of this kind were proposed by Restle (1962), Bower and Trabasso (1964; see also Trabasso and Bower 1968), Levine (1966, 1969) and others (see Brown (1974) for a review of this work). This hypothesis-testing theory has been tested quite throroughly in the case of the standard task and has survived these tests remarkably well (Brown 1974; Levine 1969, 1975; Trabasso and Bower 1968).

One important limitation of these models however is that they have been developed only for the standard concept identification task. Almost no attempt has been made to extend hypothesis-testing theory to other experimental paradigms. In this paper we will present a general framework for concept identification based on hypothesis-testing theory in which mathematical models can be developed in a relatively straightforward manner for any kind of task (including the ones described earlier). This extension of the theory is important because it is a much more stringent test of the theory.

The present theory is a modification of the duoprocess theory as presented by Chumbley (1972). Both Chumbley's original version of the duoprocess theory and the present one are extensions of the Bower and Trabasso (1964) model for concept learning. The duoprocess theory assumes that two processes may be distinguished in the learning of conceptual classifications:

(a) *a dimension-selection process* (to find the relevant dimension(s)), and
(b) *an associative process* (to learn which category goes with which value(s) on the selected dimension(s)).

These two processes are assumed to be hierarchically related all-or-none processes. It is assumed that a subject selects a new hypothesis or dimension after 'infirming errors'. An infirming error is defined as the result of one of the following two events:

(1) A stimulus is presented which the subject has associated with a response-category, and this response is not correct.
(2) The correct response is a response which is associated with one of the values on the selected dimension (or with one of the combinations of values on the selected dimensions, *e.g.* in conjunctive rules), and the stimulus that was presented did not have that value (or that combination).

The difference between our model and Chumbley's model lies in this definition of an infirming error. Chumbley (1972: 19) defines an infirming error as an error that disconfirms the first learned association (the 'hypothesis response' in his terminology), the association that is learned upon selection of a new hypothesis concerning the relevant dimension(s). In our model every error that disconfirms any of the learned associations is an infirming error. Thus, in our model the subject is assumed to behave somewhat more efficiently since he will sooner reject an incorrect hypothesis.

This modification of Chumbley's duoprocess theory has rather drastic consequences on the mathematical structure of the model. Moreover, Chumbley's analysis leads to inconsistencies when applied to conjunctive concept identification tasks. This is a consequence of the fact that his process model does not specify what the subject does when he gets a feedback that does not contradict the hypothesis response, but that does contradict one of the other learned associations. Chumbley (1972: 23) represented his version of the duoprocess theory as an absorbing Markov chain. For example, in the case of the four-choice concept problem the transition matrix is given by eq. (1) (where the error and correct states have been collapsed).

$$
\begin{array}{c|c}
 & \begin{array}{cccc} \mathrm{L} & \mathrm{PAL}_2 & \mathrm{PAL}_1 & \mathrm{AI} \end{array} \quad \mathrm{P(C)} \\
\begin{array}{c} \mathrm{L} \\ \mathrm{PAL}_2 \\ \mathrm{PAL}_1 \\ \mathrm{AI} \end{array} &
\begin{bmatrix} 1 & 0 & 0 & 0 \\ a/2 & 1-a/2 & 0 & 0 \\ 0 & 3a/4 & 1-3a/4 & 0 \\ 0 & 0 & qic & 1-qic \end{bmatrix}
\begin{bmatrix} 1 \\ 3/4 \\ 1/2 \\ 1-q \end{bmatrix}
\end{array}
\qquad (1)
$$

$$\mathrm{I.V.} = [0 \quad 0 \quad 0 \quad 1]$$

In this equation AI is the state prior to selection of the relevant pair of dimensions, PAL_1 and PAL_2 are the states where the subject is in when he has selected the two relevant dimensions and has formed resp. one or two associations, and L corresponds to the state where learning has been completed (note that PAL_3 and PAL_4 are combined into a single state L, since these two states cannot be distinguished). a refers to the probability of learning a particular value-response association, q to the probability of an error in state AI, i to the conditional probability that an error is an infirming error, and c refers to the probability of selecting the relevant pair of dimensions. The difficulty with this model is that the probability of an error in state AI (q) is not independent of the trialnumber (given the assumptions of the duoprocess theory).

Moreover, Chumbley's version of the duoprocess theory specifies that prior to selection of the relevant pair the subject may hold either a hypothesis based on two irrelevant dimensions or a hypothesis based on one relevant dimension. It can be easily shown however that these two states cannot be lumped into a single AI-state (see Greeno and Steiner (1964) for a definition of lumpability). Thus, eq. 1 is not a correct representation of the process model specified by the duoprocess theory.

Both Chumbley's version of the duoprocess theory and the present version assume that after non-infirming errors (the kind of errors that may be made after selection of the relevant dimension(s)) the subject does not resample. The present theory assumes than when the subject selects a dimension, he forms a locally consistent (partial) hypothesis. This hypothesis specifies one or more dimensions as the relevant one(s) and assigns the response-category specified by the just received feedback to the current value on the selected dimension. For example, suppose that the stimulus presented is a yellow square, the correct category is B, the subject makes an infirming error, and he selects color as the hypothesized solution dimension. In that case it is assumed that he will assign category 'B' to the dimension-value 'yellow'. Thus, his (partial) hypothesis is locally consistent (Gregg and Simon 1967): it is consistent with the just obtained feedback.

Learning of the remaining associations is assumed to occur in an all-or-none manner, with constant probability of learning on every trial where an "unlearned" stimulus (*i.e.* a stimulus for which the current hypothesis does not specify a particular response-category) is presented. If a stimulus is presented which is associated with a response-category, the subject will give that response. Otherwise he will select at random one of the "unlearned" response-categories (*i.e.* response-categories which have not yet been associated to one of the values on the selected dimension). Finally it is assumed that the probability of selecting a particular dimension is, as in the original Bower and Trabasso model, determined by the weight (or attention value) of that dimension. Fig. 1

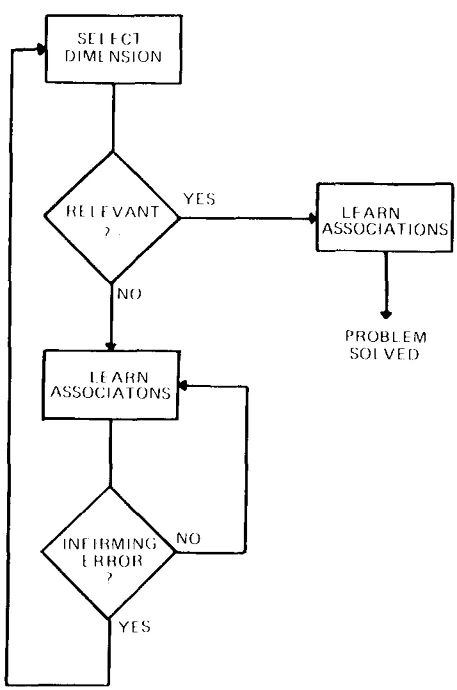

Fig. 1. A flowchart describing the proposed processing stages in complex concept identification tasks.

gives a schematic depiction of the proposed processing stages.

This may be clarified by an example. Suppose that subjects must learn the following rule:

red $\rightarrow$ 1 yellow $\rightarrow$ 3

green $\rightarrow$ 2 blue $\rightarrow$ 4

It is assumed in this and all the following cases that the subjects know the type of rule, that is, they are instructed that every value on the relevant dimension is paired with exactly one response-category. At the start of the task, the subject receives the following information: red circle $\rightarrow$ 1. He now selects "form" as the relevant dimension and forms the tentative (locally consistent) hypothesis: circle $\rightarrow$ 1. The next stimulus is a yellow triangle. The subject now guesses 2, 3 or 4. In this case he may make an error (with probability 2/3), but that does not signify that his dimension was not the correct or relevant one. After receiving feedback he learns with probability a the association triangle $\rightarrow$ 3. Suppose the next stimulus is red square. The subject guesses 2 or 4 and receives the information red square $\rightarrow$ 1. Now he has made an infirming error: he knows that circles are 1 and since the rule is one-one, squares cannot be 1 too. Another possibility for an infirming error is when the stimulus presented is a blue triangle. In that case, he would make the response '3' and get the feedback '4'. This would also tell him that his hypothesis cannot be correct since triangles cannot be both 3 and 4. In both cases he would select a new dimension and start all over again.

This revised version of the duoprocess theory extends hypothesis-testing theory to many, more complex, concept identification tasks. In particular, it generates models for all of the concept identification tasks discussed earlier. It should be noted that the duoprocess theory reduces to the Bower and Trabasso model when applied to the simple, standard concept learning task (see also Cotton 1971, 1974). It differs from several other concept-identification models for the standard concept task in that it is assumed that dimensions are sampled, not fully specified hypotheses (*i.e.* color is sampled, not the hypothesis yellow $\rightarrow$ 1, red $\rightarrow$ 2). In the simple task, dimensional sampling is equivalent to the condition known as 'local consistency' (Gregg and Simon 1967). It should be noted however that our assumption of dimension-sampling is consistent with the results reported by Gumer and Levine (1971) and Gholson and O'Connor (1975).

In the remainder of this paper we will show how specific mathematical models may be derived from this general framework for specific experimental tasks. We will also present some experimental data supporting the theory.

Experiment 1: affirmative rules

The purpose of this experiment was to separate experimentally the two processes proposed by the duoprocess theory, dimension-selection and association-learning. There were three conditions in this experiment. The first condition was a pure dimension-selection task, the second a pure associative learning task, while the third task required both dimension-selection and association-learning. We will first describe the tasks and the mathematical models derived from the duoprocess theory for those tasks.

(a) Condition 1

In condition 1 *S*s had to solve a simple concept identification problem, *i.e.* a two-category task with binary dimensions where the solution is based on a single dimension. The duoprocess theory specifies that the Markov chain given in eq. (2) describes the error-correct sequences. It is a slight modification of the Bower and Trabasso model, due to the fact that on the initial trial (trial 0) no response had to be given. It is assumed that *S*s select their first dimension at the end of this initial trial.

$$\begin{array}{c c} & \begin{array}{ccc} \mathrm{L} & \mathrm{IS} & \mathrm{IE} \end{array} \\ \begin{array}{c} \mathrm{L} \\ \mathrm{IS} \\ \mathrm{IE} \end{array} & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ c & (1-c)\frac{1}{2} & (1-c)\frac{1}{2} \end{array}\right] \end{array} \tag{2}$$

$$\mathrm{I.V.} = [c \quad (1-c)\tfrac{1}{2} \quad (1-c)\tfrac{1}{2}],$$

where L is the state where the *S* has selected the relevant dimension and makes no more errors, and IS (IE) is the state where the *S* has selected an irrelevant dimension and makes a correct (wrong) response. In this case, every error is an infirming error, so that a new dimension is sampled after every error. As in the Bower and Trabasso theory it is assumed that c, the probability of sampling the relevant dimension, is equal to the relative weight of that dimension:

$$c = \frac{w_r}{\Sigma\, w_i} \tag{3}$$

This model for the simple, standard concept task was analyzed and tested extensively by Bower and Trabasso (1964) and Trabasso and Bower (1968), with generally favorable results. Predictions for relevant statistics may be found in several sources (*e.g.* Polson 1970).

(b) Condition 2

In the second condition *S*s were presented a kind of paired-associate learning task. Every dimension had four values on it and there were four response-categories. *S*s were told which dimension was the relevant one, so they only had to find out which value on that dimension was paired with which response-category. According to the duoprocess theory described above the following Markov model should be appropriate for this task:

$$\begin{array}{c} \\ \mathrm{L} \\ \mathrm{R_2S} \\ \mathrm{R_2E} \\ \mathrm{R_1S} \\ \mathrm{R_1E} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{L} & \mathrm{R_2S} & \mathrm{R_2E} & \mathrm{R_1S} & \mathrm{R_1E} \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \frac{a}{3} & (1-\frac{a}{3})\frac{3}{4} & (1-\frac{a}{3})\frac{1}{4} & 0 & 0 \\ a & (1-a)\frac{3}{4} & (1-a)\frac{1}{4} & 0 & 0 \\ 0 & \frac{3a}{8} & \frac{a}{8} & (1-\frac{a}{2})\frac{1}{2} & (1-\frac{a}{2})\frac{1}{2} \\ 0 & \frac{3a}{4} & \frac{a}{4} & (1-a)\frac{1}{2} & (1-a)\frac{1}{2} \end{array}\right] \end{array} \tag{4}$$

$$\text{I.V.} = [0 \quad 0 \quad 0 \quad \tfrac{1}{2} \quad \tfrac{1}{2}].$$

In eq. 4, R_jS (R_jE) is the state where j associations have been learned and a correct (wrong) response is made. In state L no more errors are made. Note that our assumptions imply that this will be the case as soon as three associations have been learned. In eq. 4, a denotes the probability that an association will be learned given an opportunity for learning. Of course in this case none of the errors is infirming. Comparison of eq. 4 with eq. 1 shows that after collapsing of error and correct states this model is the same as that proposed by Chumbley (1972) once the AI-state has been left.

(c) Condition 3

Conditions 1 and 2 may be regarded as experimental tasks that single out the two processes assumed by the duoprocess theory. These tasks are "combined" in condition 3. In this task the problem is to categorize correctly a series of stimuli which vary on four-valued dimensions into four categories. The categorization rule is based on a single dimension and there is a one-one relation between the values of the relevant dimension and the response categories.

After selection of the relevant dimension the model for condition 2 applies. Prior to selection of the relevant dimension the *S* is processing an irrelevant dimension. He may have associated one to four values on that irrelevant dimension with a response-category. Thus, when the *S* holds an irrelevant dimension, he may be in one of three states, I_1, I_2, or I_4, where the subscript indicates the number of associations learned (states I_3 and I_4 are combined since the choice behavior of the *S* is the same in both states). Eq. (5) gives the Markov model specified by our version of the duoprocess theory for this condition. For reasons of clarity we have collapsed the error and correct states. (For more details we refer to Raaijmakers 1976.)

$$
\begin{array}{c}
\\ L \\ R_2 \\ R_1 \\ I_4 \\ I_2 \\ I_1
\end{array}
\begin{array}{c}
\begin{array}{cccccc} L & R_2 & R_1 & I_4 & I_2 & I_1 \end{array} \\
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{a}{2} & 1-\frac{a}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{3a}{4} & 1-\frac{3a}{4} & 0 & 0 & 0 \\
0 & 0 & \frac{3c}{4} & \frac{1}{4} & 0 & (1-c)\frac{3}{4} \\
0 & 0 & \frac{5c}{8} & \frac{a}{4} & (\frac{3}{8}-\frac{a}{4}) & (1-c)\frac{5}{8} \\
0 & 0 & \frac{3c}{8} & 0 & \frac{9a}{16} & 1-\frac{3c}{8}-\frac{9a}{16}
\end{array}\right]
\end{array}
\begin{array}{c}
P(C) \\
\left[\begin{array}{c} 1 \\ \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{array}\right]
\end{array}
\tag{5}
$$

$$\text{I.V.} = [0 \quad 0 \quad c \quad 0 \quad 0 \quad 1-c].$$

The analysis of the model given in eq. 5 is somewhat more complicated than the relatively simple analysis of Chumbley's version of the duoprocess theory (Chumbley 1972: appendix) due to the fact that the I-states cannot be lumped into a single state, as in Chumbley's model (see eq. 1). This is a consequence of our assumptions concerning infirming errors which imply that the probability of an error being an infirming error is not constant during all stages of processing an irrelevant dimension. Explicit expressions for the probability distributions of the various statistics are hard to obtain. As shown in Appendix A, it is possible to derive analytic solutions for the probability generating functions (Feller 1968) for the total number of errors and the trial of last error and thus for the means and variances of those statistics. The latter may be used for parameter estimation purposes. Given estimates for the parameters other predicted statistics may be derived using the general numerical methods given by Millward (1969).

Method

Materials

The stimuli were geometric figures drawn on cards of 24 X 30 cm. There were four dimensions. Depending on the experimental conditions there were two (condition 1) or four (conditions 2 and 3) values on each dimension. The dimensions were: (a) position of dot (upper right-, lower right-, lower left-, or upper left-hand corner of the card), (b) "texture" of the figures (black, white, white with horizontal black stripes, or white with vertical black stripes), (c) number of figures on a card (one, two, three, or four) and (d) form (square, triangle, circle, or cross).

Apparatus and procedure

The experiment was run on a PDP 11/45 computer. *S*s were seated in front of a

T.V. monitor and a panel with a number of response-buttons, four of which were used in this experiment. Underneath each of these four buttons was a category-label (A, B, C, or D). *S*s responded by pressing the appropriate button. The stimuli were recorded and stored on a videodisc-recorder (AMPEX MD 400). For condition 1 a random sequence of 50 pictures was recorded, for conditions 2 and 3 a sequence of 250 pictures.

*S*s participated in groups of one to four persons. Curtains were placed between them so they could not watch each other. The groups had different starting points in the sequence and different stepsizes. Therefore, the stimulus sequence can be considered to be 'random' (at least as random as was possible with the apparatus used). The three conditions were presented in the same order for all *S*s, namely first condition 1, then condition 3 and next condition 2. *S*s learned to a criterion of ten consecutive correct responses. There was no time-limit for responding, although they were instructed not to wait too long (not longer than about 10 sec). After the *S* had given his response, he was shown the stimulus with the correct category letter written underneath. This feedback was presented for 3 sec. There was an intertrial interval which lasted until 3 sec after the last *S* (of that group) had received feedback. Prior to the first trial the *S*s were shown a card with the appropriate feedback (for 3 sec). After an interval of 4 sec the first to be categorized stimulus was presented.

Before each condition *S*s were given instructions as to the type of problem and the number and kind of dimensions involved. Two groups of *S*s were formed, depending on which of the dimensions was relevant, "position of dot" (dim 1) or "texture" (dim 2). The first group had dim 1 as the relevant dimension in conditions 1 and 2, while the second group had dim 2 as the relevant dimension in those tasks. In condition 3 the relevant dimensions were reversed so that group 1 now had dim 2 as the relevant dimension and group 2, dim 1.

Subjects

A total of 92 *S*s participated in this experiment. About 60 of them were paid volunteers from a local secondary school. The remainder were students who served as a course requirement, and a few employees from the University of Nijmegen. The age of the *S*s varied between 16 and 30 years. None of the *S*s was practiced in concept identification tasks. Five protocols were deleted from the analysis. In four cases (all stemming from condition 3) the problem was not solved before or on trial 70, probably due to a misunderstanding of the instructions. In the fifth case the *S* told the experimenter that she had misinterpreted the instruction for condition 2.

Results and discussion

The results are presented here as two parallel series of experiments (see table 1). The reasons for presenting the results in this way are as follows:

Table 1
Subjects and relevant dimensions for each of the two series.

	Condition		
	1	2	3
Series 1	Group 1 Dim 1	Group 2 Dim 2	Group 2 Dim 1
Series 2	Group 2 Dim 2	Group 1 Dim 1	Group 1 Dim 2

(1) We assume that the parameter c, the probability of sampling the relevant dimension, depends most on which dimension is relevant. Thus, we should have the same dimensions relevant in conditions 1 and 3.
(2) We assume that the parameter a, the probability of learning a value-category association, does not depend on which of the dimensions is relevant, but may be influenced by individual differences. Therefore, we should have the same subjects in conditions 2 and 3.

It may be remarked that there is a slight inconsistency here: we assume in the application of the model that for a given group of subjects the parameter a is constant for all subjects, although we do admit possible interindividual differences. Nothing is known, however, about the effect of individual differences in this parameter on the detailed predictions of the model. If there would be an interaction between a and c the above design would be somewhat inadequate.

It may be the case that the parameter a is also influenced by the complexity of the task, *i.e.* if the subject tries to remember more things at the same time (*e.g.* which dimensions he has already rejected) this might have an influence on the association parameter. For these reasons we will estimate a separately for conditions 2 and 3. It will then be possible to decide *a posteriori* whether these two values (denoted by $\hat{a}_2$ and $\hat{a}_3$) can be considered as estimates of the same "true" parameter. Thus, three parameters were estimated for each series: c, a_2, and a_3.

In table 2 are given the observed and predicted values of several summary statistics for series 1 and in table 3 for series 2. The parameter estimates indicated at the bottom of these tables were obtained by minimizing the sum of the squared deviations of the observed from the

Table 2
Summary statistics for series I.

	Condition 1		Condition 2		Condition 3	
	Observed	Predicted	Observed	Predicted	Observed	Predicted
Total number of errors, T						
Mean	1.40 [a]	1.41	1.82 [a]	1.97	4.78 [a]	5.12
s.d.	1.30	1.85	1.51	1.27	3.82	3.75
Trial of last error, L						
Mean	2.91 [a]	2.83	4.31 [a]	4.19	9.67 [a]	9.44
s.d.	3.37	4.06	3.51	3.40	8.15	6.68
Numbers of errors before the first success, J						
Mean	0.49	0.41	0.84	0.73	1.33	1.56
s.d.	0.75	0.77	1.22	0.90	1.72	1.84
Mean total number of error runs, R	1.11	1.00	1.51	1.49	2.87	2.70
Mean number of error runs of length j, r_j						
r_1	0.85	0.71	1.36	1.11	1.91	1.54
r_2	0.21	0.21	0.11	0.29	0.56	0.58
r_3	0.04	0.06	0.0	0.07	0.13	0.27
r_4	0.0	0.02	0.02	0.01	0.09	0.14
Number of subjects	47		45		45	

[a] Used to estimate the parameters: $\hat{c} = 0.414$, $\hat{a}_2 = 0.591$, and $\hat{a}_3 = 0.425$

Table 3
Summary statistics for series II.

	Condition 1		Condition 2		Condition 3	
	Observed	Predicted	Observed	Predicted	Observed	Predicted
Total number of errors, T						
Mean	3.84 [a]	4.00	1.43 [a]	1.62	6.70 [a]	7.96
s.d.	4.43	4.47	0.91	1.01	5.93	7.25
Trial of last error, L						
Mean	8.49 [a]	8.00	3.39 [a]	3.26	12.56 [a]	11.79
s.d.	10.45	9.38	2.70	2.67	10.56	10.04
Number of errors before the first success, J						
Mean	0.69	0.67	0.43	0.69	1.74	2.07
s.d.	1.02	1.05	0.58	0.82	2.17	2.41
Mean total number of error runs, R	2.56	2.40	1.24	1.27	3.47	3.08
Mean number of error runs of length j, r_j						
r_1	1.78	1.44	1.09	0.99	1.88	1.36
r_2	0.44	0.58	0.11	0.24	0.81	0.64
r_3	0.20	0.23	0.04	0.04	0.42	0.38
r_4	0.09	0.09	0.0	0.01	0.12	0.24
Number of subjects	45		46		43	

[a] Used to estimate the parameters: $\hat{c} = 0.200$, $\hat{a}_2 = 0.721$, and $\hat{a}_3 = 0.694$.

predicted values divided by the predicted values for the mean of the total number of errors and the trial of last error for all three conditions. The relevant formula's are given in appendix A. The other predictions were obtained with a computer program based on the numerical methods described in Millward (1969).

As can be seen from these tables the predictions agree reasonably well with the data. For both series and each condition, the difference between the observed and the predicted distributions of the total number of errors and the trial of last error was tested with the Kolmogorov-Smirnov statistic D (the maximum deviation of the observed from the predicted distribution function). None of these differences was significant at $\alpha = 0.10$. In view of the fact that only three parameters were estimated for each series and that 12 tests were carried out (six for each series), the results are quite satisfactory.

It may be noted that there is a strong effect of differential salience of the dimensions which overshadows any effect of differences in the association parameter. By comparing conditions 1 and 3 of series 1 and 2 it becomes obvious that the dimension "position of dot" (dim 1) was much more salient than the dimension "texture" (dim 2). In the first series the parameters a_2 and a_3 seem to have different "true" values. The standard error in these estimates is about 0.05 when $n \approx 45$ (this was found in simulations of the model). However this is not true for the second series.

The present model may be easily generalized to other types of sampling rules. In fact, due to the analytical separation of the dimension-selection and the associative learning processes, it is easy to generalize any sampling scheme developed in the context of the standard, two-category task to the present task as long as the sampling scheme assumes that at any stage only a single dimension is held. For example, instead of assuming that dimension sampling is with replacement, we might assume that the subject samples without replacement of the just infirmed dimension (local nonreplacement). If we denote by c_0 the probability of sampling the relevant dimension on the initial trial, a good approximation is obtained by setting c in the transition matrix equal to $c_1 = (m - 1)\, c_0/(m - 2 - c_0)$ where m is the number of dimensions. The expected number of irrelevant dimensions sampled prior to selection of the relevant dimension is then given by $E(N) = (1 - c_0)/c_1$. This approximation is exact if all dimensions have equal weights. Similar methods may be applied if one makes still other assumptions regarding the dimension-selection process, for example "global nonreplacement", the permanent elimination of rejected dimensions (Gregg and Simon 1967).

The present model may be used in the analysis of the effect of various task variables on performance in concept identification tasks. For example, it may be used to analyze the effect of the type of feedback given (see Comstock and Chumbley 1973). Note finally that the present model is consistent with the results reported by Polson and Dunham (1971). In the next section we will show how the same theory may be used to generate models for the behavior of subjects in conjunctive concept identification tasks.

Experiment 2: the four-choice concept problem

In this section we will apply our revised version of the duoprocess theory on one type of conjunctive rule, the so-called four-choice concept problem. In the four-choice concept problem stimuli are presented which vary on $K + 2$ binary dimensions, two independent relevant dimensions and K irrelevant dimensions. The correct classification is determined by the combination of the values of the two relevant dimension. To every combination there corresponds one response-category. For example, if color (blue or yellow) and form (triangle or square) are relevant, the classification rule might be: all blue triangles in category A, all yellow triangles in category B, all yellow squares in category C, and all blue squares in category D. This particular type of concept task has been studied by, among others, Bourne and Haygood (1959), Bourne and Restle (1959), Trabasso and Bower (1964), and Wandmacher and Vorberg (1974).

It is assumed that the S forms a hypothesis as to which dimensions are relevant, by sampling, in a successive manner, two dimensions. As in the model for affirmative rules the selection probabilities depend on the relative weights of the dimensions. The probability of sampling the dimensions i and j, c_{ij}, equals:

$$c_{ij} = r_i\left(\frac{r_j}{1 - r_i}\right) + r_j\left(\frac{r_i}{1 - r_j}\right), \tag{6}$$

where r_i and r_j are the relative weights of dimensions i and j in the set of $K + 2$ dimensions. If we assume that the two relevant dimensions are equally salient, the probability of selecting the pair of relevant dimensions is given by:

$$c_1 = 2r^2/(1 - r), \tag{7}$$

where r denotes the relative weight of the relevant dimension. Instead of the pair of relevant dimensions the S may also sample one relevant and one irrelevant dimension or two irrelevant dimensions. The probabilities of these two types of hypotheses are given by resp. c_2 and c_3:

$$c_2 = 2r(1 - 2r)[1/(1 - r) + K/(K - 1 + 2r)] \tag{8}$$

$$c_3 = 1 - c_1 - c_2 = (K - 1)(1 - 2r)^2/(K - 1 + 2r). \tag{9}$$

It should be noted that the assumption of equal weights for the two relevant dimensions does not affect the predictions very much. For example, if $r_1 = 0.40$ and $r_2 = 0.20$ and $K = 2$, then we get $c_1 = 0.23$, $c_2 = 0.67$, and $c_3 = 0.10$. Assuming $r_1 = r_2 = r$ and setting $r = 0.286$ gives $c_1 = 0.23$, $c_2 = 0.65$, and $c_3 = 0.12$. The duoprocess framework does not of course necessitate the assumption that the dimensions are sampled successively. Instead it might be assumed that a pair of dimensions is selected from the set of dimension-pairs. It is easy to derive similar equations for such a sampling assumption (for more details see Raaijmakers 1977). If all dimensions have equal weights these two methods of sampling are equivalent, that is, they give the same results for c_1, c_2, and c_3.

In all other respects the development of a model for this type of conjunctive task proceeds in a similar way as for the affirmative tasks. The duoprocess theory specifies that the S may be in three types of states: RR_j, the state where he has selected the two relevant dimensions and has learned j associations; IR_j, the state where he has selected one relevant and one irrelevant dimension and has learned j associations; and II_j, where he has selected two irrelevant dimensions and has learned j associations between the combinations of values from those dimensions and the response-categories. As in the case of affirmative rules our assumptions imply that certain states may be lumped since the behavior of the S is the same in those states. Thus, the states RR_3 and RR_4 are combined to a single state (L), as are the states IR_3 and IR_4 (both denoted by IR_4) and the states II_3 and II_4 (denoted by II_4). The state IR_2 however must be subdivided in the two states IR_{21} and IR_{22}. This is because the two learned associations may involve either only one of the two values of the relevant dimension (IR_{21}) or both values on that dimension (IR_{22}), and because the probability of an infirming error differs in these two cases. In this way we obtain the Markov model given in eq. 10 (see page 250).

From eq. 10 is obvious that the behavior in the two sets of states RR_j and II_j is equivalent to the behavior in the corresponding states R_j and I_j of the model for the affirmative task given in eq. 5. We will make use of this correspondence between the two models in the analysis of the present model. As shown in appendix B, it is again possible to derive predictions for the means and variances of the statistics TNE and TLE which may be used for parameter estimation purposes.

We will compare the above model with the model proposed for this task by Trabasso and Bower (1964) which is based on the so-called subproblem learning theory. In this theory it is assumed that the S solves the problem by independently solving two subproblems. In the example given earlier the subproblems would be: (1) blue in A or D, yellow in B or C, and (2) triangle in A or B, square in C or D. Trabasso and Bower (1964) assumed that Ss could solve each of these subproblems on every trial in an all-or-none fashion, whether an error was made on that trial or not. It is possible however to adapt the theory to make it more compatible with the Bower and Trabasso model for simple concept identification tasks, in which it is assumed that Ss solve simple problems by selecting a new hypothesis only after errors.

$$
\begin{array}{c}
 \\ \mathrm{L} \\ \mathrm{RR}_2 \\ \mathrm{RR}_1 \\ \mathrm{IR}_4 \\ \mathrm{IR}_{21} \\ \mathrm{IR}_{22} \\ \mathrm{IR}_1 \\ \mathrm{II}_4 \\ \mathrm{II}_2 \\ \mathrm{II}_1
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
\mathrm{L} & \mathrm{RR}_2 & \mathrm{RR}_1 & \mathrm{IR}_4 & \mathrm{IR}_{21} & \mathrm{IR}_{22} & \mathrm{IR}_1 & \mathrm{II}_4 & \mathrm{II}_2 & \mathrm{II}_1
\end{array} \\
\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a}{2} & 1-\frac{a}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3a}{4} & 1-\frac{3a}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}c_1 & \frac{1}{2} & 0 & 0 & \frac{1}{2}c_2 & 0 & 0 & \frac{1}{2}c_3 \\
0 & 0 & \frac{1}{4}c_1 & \frac{a}{2} & (\frac{3}{4}-\frac{a}{2}) & 0 & \frac{1}{4}c_2 & 0 & 0 & \frac{1}{4}c_3 \\
0 & 0 & \frac{1}{2}c_1 & \frac{a}{4} & 0 & (\frac{1}{2}-\frac{a}{4}) & \frac{1}{2}c_2 & 0 & 0 & \frac{1}{2}c_3 \\
0 & 0 & \frac{1}{4}c_1 & 0 & \frac{a}{8} & \frac{a}{2} & (\frac{3}{4}-\frac{5a}{8}+\frac{1}{4}c_2) & 0 & 0 & \frac{1}{4}c_3 \\
0 & 0 & \frac{3}{4}c_1 & 0 & 0 & 0 & \frac{3}{4}c_2 & \frac{1}{4} & 0 & \frac{3}{4}c_3 \\
0 & 0 & \frac{5}{8}c_1 & 0 & 0 & 0 & \frac{5}{8}c_2 & \frac{a}{4} & (\frac{3}{8}-\frac{a}{4}) & \frac{5}{8}c_3 \\
0 & 0 & \frac{3}{8}c_1 & 0 & 0 & 0 & \frac{3}{8}c_2 & 0 & \frac{9a}{16} & (\frac{5}{8}-\frac{9a}{16}+\frac{3}{8}c_3)
\end{array}\right]
\end{array}
\begin{array}{c}
\mathrm{P(C)} \\
\left[\begin{array}{c}
1 \\ \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{8} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4}
\end{array}\right]
\end{array}
\qquad (10)
$$

$$\mathrm{I.V.} = [0 \quad 0 \quad c_1 \quad 0 \quad 0 \quad 0 \quad c_2 \quad 0 \quad 0 \quad c_3].$$

Suppose that the S solves each of the subproblems in the way described by the Bower and Trabasso (1964) model. That is, he selects hypotheses separately and independently for each subproblem, and he rejects a hypothesis if the outcome of a trial is inconsistent with that hypothesis. In the case that the S holds two irrelevant hypotheses, the probability that an error is infirming to each of the two hypotheses can be calculated as follows. The S makes an error if he chooses, in accordance with his irrelevant hypothesis, any of the three incorrect response-categories. The outcome is inconsistent with both hypothesis in one of these three cases. In the other two it is inconsistent with only one hypothesis.

Following Trabasso and Bower, we will distinguish three states 0, 1, and 2, depending on whether 0, 1, or 2 subproblems have been solved. In the states 0 and 1 a distinction is made whether a correct response or an error is made. Let c denote the probability of sampling the relevant hypothesis for a particular subproblem after an inconsistent outcome. The probability of going after an error in state 0E to states 2, 1, or 0 is given by:

$$P(2 \text{ on trial } n+1 \mid 0\text{E on trial } n) = \tfrac{1}{3}c^2,$$

$$P(1 \text{ on trial } n+1 \mid 0\text{E on trial } n) = \tfrac{2}{3}c + \tfrac{1}{3}[2c(1-c)],$$

$$P(0 \text{ on trial } n+1 \mid 0\text{E on trial } n) = \tfrac{2}{3}(1-c) + \tfrac{1}{3}(1-c)^2.$$

This leads to the following transition matrix for the Trabasso and Bower model with the 'learning-on-errors' assumption:

$$\begin{array}{c@{\;}c} & \begin{array}{ccccc} \;2\; & \;1\text{S}\; & \;1\text{E}\; & \;0\text{S}\; & \;0\text{E}\; \end{array} \\ \begin{array}{c} 2 \\ 1\text{S} \\ 1\text{E} \\ 0\text{S} \\ 0\text{E} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ c & \frac{1}{2}(1-c) & \frac{1}{2}(1-c) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3}c^2 & \frac{c}{3}(2-c) & \frac{c}{3}(2-c) & \frac{1}{12}(1-c)(3-c) & \frac{1}{4}(1-c)(3-c) \end{bmatrix} \end{array} \qquad (11)$$

$$\text{I.V.} = [0 \quad 0 \quad 0 \quad \tfrac{1}{4} \quad \tfrac{3}{4}].$$

If we combine the error and correct states, the following transition matrix is obtained:

$$\begin{array}{c@{\;}c@{\;}c} & \begin{array}{ccc} 2 & 1 & 0 \end{array} & \text{P(C)} \\ \begin{array}{c} 2 \\ 1 \\ 0 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ \frac{c}{2} & 1-\frac{c}{2} & 0 \\ (\frac{c}{2})^2 & 2(\frac{c}{2})(1-\frac{c}{2}) & (1-\frac{c}{2})^2 \end{bmatrix} & \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \end{array} \qquad (12)$$

$$\text{I.V.} = [0 \quad 0 \quad 1].$$

This matrix is equal to the one given by Trabasso and Bower (1964: 157) after substitution of $\theta = c/2$. Thus, a similar relationship holds between these two models as between the Bower and Trabasso model for concept learning and the one-element model: the two models do not make very different predictions for the ordinary experiment.

It should be noted that the revised model implies the socalled multiplication rule, just as was the case with the original model. That is, the probability of a correct response (P_n) is equal to the product of the probabilities of a correct response to each of the subproblems (P'_n):

$$P_n = 1 - \left(1 - \frac{c}{2}\right)^{n-1} + \frac{1}{4}\left(1 - \frac{c}{2}\right)^{2(n-1)}$$
$$= \left[1 - \frac{1}{2}\left(1 - \frac{c}{2}\right)^{n-1}\right]^2 = (P'_n)^2 \tag{13}$$

Results consistent with this subproblem learning theory were obtained by Trabasso and Bower (1964) and Wandmacher and Vorberg (1974). It may be the case however that these results can be just as well explained by the duoprocess theory (see Chumbley 1972). One problem with the Trabasso and Bower data is that in their experiment the stimuli were not presented in a random order. In the procedure they used, no two successive stimuli belonged to the same category. This probably has an effect on the trial of last error which is not incorporated in their model (see Cotton (1971, 1974) for a discussion of the role of stimulus sequence on concept identification). Results consistent with a duoprocess interpretation were obtained by Overstreet and Dunham (1969) and Thomson (1972).

Method

Materials

The same set of stimuli was used as in the first experiment. The two relevant dimensions were: "position of dot" (upper left or lower right corner) and "form" (triangle or square). The two irrelavant dimensions were: "texture" (black or white with horizontal stripes) and "number" (two or three figures on a card).

Apparatus and procedure

The same apparatus and general procedure were used as in experiment 1. *S*s were given only a brief instruction describing the type of rule and the dimensions used.

Subjects

The same group of 92 *S*s as in the first experiment participated in this experiment which took place immediately after the first experiment. The protocols of 4 *S*s could not be used due to apparatus failure. Therefore, all analyses are based on the data of the 88 remaining *S*s.

Results and discussion

In table 4 are given the observed values of several summary statistics together with the corresponding values predicted by, respectively, the present version of the duoprocess theory (D theory) and the revised Trabasso and Bower model (T-B theory). The two parameters of the duoprocess theory, a and r, were estimated by minimizing the sum of the squared deviations of the observed from the predicted values divided by the predicted values for the means and variances of the total number of errors and the trial of last error. The parameter c for the T-B model was estimated from the mean total number of errors. The remaining predictions were obtained with a computer program based on the methods developed by Millward (1969).

As can be seen in table 4, the values predicted by the duoprocess theory are quite close to the observed values. Only the observed vari-

Table 4
Summary statistics for experiment 2.

	Observed	Predicted D theory	Predicted T-B theory
Total number of errors, T			
Mean	8.72	9.64 [a]	8.72 [b]
s.d.	7.58	7.93 [a]	5.67
Trial of last error, L			
Mean	17.56	17.45 [a]	14.80
s.d.	14.50	13.35 [a]	10.66
Number of errors before the first success, J			
Mean	1.50	1.58	2.05
s.d.	1.58	1.91	2.05
Mean total number of error runs, R	4.83	4.55	4.12
Mean number of error runs of length j, r_j			
r_1	2.83	2.32	1.98
r_2	1.00	1.02	1.01
r_3	0.55	0.53	0.52
r_4	0.25	0.29	0.28

[a] Used for parameter-estimation: $\hat{r} = 0.288$, $\hat{a} = 0.364$
[b] Used for parameter-estimation: $\hat{c} = 0.199$

ance of the number of errors before the first success is significantly lower than predicted. The Trabasso and Bower theory however predicts a still higher value. The T-B theory also predicts too low values for the variance of the total number of errors ($\chi^2 = 155.5$, $df = 87$, $p < 0.001$) and the trial of last error ($\chi^2 = 161.0$, $df = 87$, $p < 0.001$). Thus, the predictions of the duoprocess theory are in general closer to the data than those of the subproblem learning theory. The Kolmogorov-Smirnov test for goodness-of-fit was applied to the distribution of the total number of errors and the trial of last error. In both cases, the duoprocess model gave a better fit. In the case of TNE, significant deviations ($\alpha = 0.05$) were observed for the T-B theory, but not for the D-theory. Thus our results are more consistent with the duoprocess theory than with the subproblem learning theory. It is interesting to note that our estimate for r agrees quite well with the corresponding parameter estimates obtained in the first experiment. In that experiment we obtained a relative weight of about 0.40 for the dimension "position of dot" (one of the relevant dimensions in this task) and about 0.20 for the dimension "texture" (now irrelevant). Assuming equal weights for the remaining two dimensions gives $r_1 = 0.40$ and $r_2 = 0.20$. As described earlier this corresponds (under the assumption $r_1 = r_2 = r$) to the value $r = 0.286$, nearly equal to the obtained estimate $\hat{r} = 0.288$.

Moreover, there are several conceptual problems with the Trabasso and Bower theory. Firstly, Wandmacher and Vorberg (1974) showed that in order to explain their results a paired-associate learning stage had to be assumed: after the subject has selected the relevant dimension for a particular subproblem, he still has to associate the response-categories in the correct way to the values on that dimension, and this cannot all be accomplished on a single trial. However, if such an assumption is made, one must also drop the assumption that resampling occurs after every error prior to selection of the relevant dimension (Wandmacher and Vorberg 1974: 221), if the theory is to remain plausible. This means that some concept of resampling only after infirming errors has to be assumed. Thus, a kind of duoprocess subproblem learning theory would be needed. Secondly, the independence assumption is, as it stands, not very plausible. As was already argued by Chumbley (1972), it is not very likely that a subject selects the same dimension as relevant for both subproblems. Thus, the subproblems cannot be completely independent.

* * *

Summarizing the results of experiments 1 and 2, it seems fair to conclude that the present version of the duoprocess theory is a promising starting point for analyzing concept identification tasks, complex as well as simple, with affirmative as well as conjunctive rules.

Appendix A

For parameter estimation purposes we will need the formula's for the predicted mean total number of errors (TNE) and the predicted mean trial of last error (TLE). For condition 1 these means are:

$$\begin{aligned} \mathrm{E}(\mathrm{TNE}_1) &= (1-c)/c \\ \mathrm{E}(\mathrm{TLE}_1) &= 2(1-c)/c \end{aligned} \tag{14}$$

For condition 2 the corresponding predictions were derived by Chumbley (1972: appendix):

$$\begin{aligned} \mathrm{E}(\mathrm{TNE}_2) &= 7/(6a) \\ \mathrm{E}(\mathrm{TLE}_2) &= 2(10+9a)/[3a(2+a)(1+a)] \end{aligned} \tag{15}$$

The analysis of the duoprocess model for condition 3 is more complicated. In the model presented in eq. 5 the occurrence of an infirming error is a transient (uncertain) recurrent event: upon the occurrence of an infirming error the subject starts anew by sampling (with replacement) a new dimension (see also fig. 1). With probability $1 - c$ he will select an irrelevant dimension and will eventually make another infirming error; with probability c he will select the relevant dimension and make no more infirming errors. Thus, the recurrence probability of an infirming error equals $(1 - c)$. It is therefore convenient to express the statistics TNE and TLE as the sum of a random number of mutually independent random variables:

$$X = V_1 + V_2 + V_3 + \ldots + V_N + R.$$

In this equation V_i stands for the number of errors made during processing of the i-th selected irrelevant dimension (in the case of TNE) or the number of trials up to and including the occurrence of an infirming error (in the case of TLE). R stands for the number of errors or the trial of last error associated with the processing of the relevant dimension. N is a random variable corresponding to the number of infirming errors, or, equivalently, to the number of times that an irrelevant dimension is sampled prior to sampling of the relevant dimension.

For a particular value of N, say n, the probability generating function (PGF) of X is equal to $[F_v(s)]^n F_r(s)$, where $F_v(s)$ and $F_r(s)$ denote the PGF's for resp. V_i and R. The unconditional PGF for X is then given by:

$$\begin{aligned} G_x(s) &= \sum_{n=0}^{\infty} P(N=n)[F_v(s)]^n F_r(s) \\ &= F_n[F_v(s)]\, F_r(s). \end{aligned} \tag{16}$$

From eq. 16 we get:

$$\mathrm{E}(X) = = \mathrm{E}(N)\,\mathrm{E}(V) + \mathrm{E}(R)$$

$$\mathrm{Var}(X) = \mathrm{Var}(N)\mathrm{E}^2(V) + \mathrm{E}(N)\,\mathrm{Var}(V) + \mathrm{Var}(R).$$

$\mathrm{E}(N)$ and $\mathrm{Var}(N)$ are determined by the recurrence probability and are equal to $\mathrm{E}(\mathrm{TNE}_1)$ and $\mathrm{Var}(\mathrm{TNE}_1)$ since in that condition $V_i = 1$ for all i and $\mathrm{E}(R) = \mathrm{Var}(R) = 0$. $\mathrm{E}(R)$ and $\mathrm{Var}(R)$ are given by the corresponding statistics derived for the model for condition 2 (see eq. 4) since in that case $\mathrm{E}(N) = \mathrm{Var}(N) = 0$. $\mathrm{E}(V)$ and $\mathrm{Var}(V)$ may be derived by applying standard methods to the following Markov chain, where the process enters the absorbing state upon the occurrence of an infirming error:

$$\begin{array}{c|c} & \begin{array}{cccc} \mathrm{A} & \mathrm{I}_4 & \mathrm{I}_2 & \mathrm{I}_1 \end{array} \\ \begin{array}{c} \mathrm{A} \\ \mathrm{I}_4 \\ \mathrm{I}_2 \\ \mathrm{I}_1 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{5}{8} & \frac{a}{4} & \left(\frac{3}{8} - \frac{a}{4}\right) & 0 \\ \frac{3}{8} & 0 & \frac{9a}{16} & \left(\frac{5}{8} - \frac{9a}{16}\right) \end{bmatrix} \end{array} \quad \begin{array}{c} \mathrm{P(C)} \\ \begin{bmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \end{array} \tag{17}$$

$$\mathrm{I.V.} = [0 \quad 0 \quad 0 \quad 1].$$

After some tedious calculations we get:

$$\begin{aligned} \mathrm{E}(\mathrm{TNE}_v) &= 2(10+3a)(1+a)/[(5+2a)(2+3a)] \\ \mathrm{E}(\mathrm{TLE}_v) &= 8(10+3a)(1+a)/[3(5+2a)(2+3a)]. \end{aligned} \tag{18}$$

$E(TNE_r)$ and $E(TLE_r)$ are given in eq. 15, and $E(N) = (1 - c)/c$. It should be noted that the above method of analysis is quite general and may be applied to any concept identification task with an affirmative rule. For example, with ternary dimensions one would obtain a Markov model with states L, R_1, I_3, and I_1, and the same method of analysis would be applicable.

Appendix B

A similar type of analysis is possible for the model presented in eq. 10. The statistics TNE and TLE can be expressed as the following sum:

$$X = V_1 + V_2 + ... + V_M + W_1 + W_2 + ... + W_N + R.$$

In this equation V_i stands for the number of errors made during processing of the i-th pair of irrelevant dimensions (in the case of TNE) or the number of trials up to and including the occurrence of an infirming error (in the case of TLE). W_j stands for the same statistics during processing of the j-th selected pair consisting of one relevant and one irrelevant dimension. R stands for the number of errors (or the trial of last error) associated with the processing of the pair of relevant dimensions. M and N are random variables corresponding to the number of times that a hypothesis based on resp. two irrelevant or one relevant and one irrelevant dimension is sampled before sampling of the solution dimensions.

For particular values of M and N, say m and n, the PGF of X is given by $[F_v(s)]^m [F_w(s)]^n F_r(s)$ where $F_v(s)$, $F_w(s)$ and $F_r(s)$ are the PGF's for resp. V_i, W_j, and R. As M and N are not constant the PGF for X is given by:

$$G_x(s) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [F_v(s)]^m [F_w(s)]^n F_r(s) P(M = m, N = n). \tag{19}$$

Let $H(s_1, s_2)$ be the bivariate probability generating function for the bivariate distribution of (m, n), *i.e.*:

$$H(s_1, s_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (s_1)^m (s_2)^n P(M = m, N = n). \tag{20}$$

Comparing eqs. 19 and 20, we see that $G_x(s)$ may be written as:

$G_x(s) = H[F_v(s), F_w(s)]\, F_r(s)$.

The mean and variance of X can be shown to be:

$$\mathrm{E}(X) \quad = \mathrm{E}(M)\,\mathrm{E}(V) + \mathrm{E}(N)\,\mathrm{E}(W) + \mathrm{E}(R) \tag{21}$$

$$\begin{aligned}\mathrm{Var}(X) = {} & \mathrm{Var}(M)\mathrm{E}^2(V) + \mathrm{E}(M)\,\mathrm{Var}(V) + \mathrm{Var}(N)\mathrm{E}^2(W) \\ & + \mathrm{E}(N)\,\mathrm{Var}(W) + 2\mathrm{E}(V)\,\mathrm{E}(W)\,\mathrm{Cov}(M, N) + \mathrm{Var}(R).\end{aligned} \tag{22}$$

In case the dimension selection is with replacement, the distribution of (m, n) is given by:

$$P(M = m, N = n) = \binom{m+n}{m} (c_3)^m\, (c_2)^n\, c_1 ,$$

$$H(s_1, s_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{m} (c_3 s_1)^m\, (c_2 s_2)^n\, c_1 .$$

By substituting $z = m + n$:

$$\begin{aligned}H(s_1, s_2) &= \sum_{z=0}^{\infty} \sum_{m=0}^{z} \binom{z}{m} (c_3 s_1)^m\, (c_2 s_2)^{z-m}\, c_1 \\ &= \sum_{z=0}^{\infty} (c_3 s_1 + c_2 s_2)^z\, c_1 \\ &= c_1/(1 - c_3 s_1 - c_2 s_2).\end{aligned}$$

For the mean, variance, and covariance of M and N we find:

$\mathrm{E}(M) \quad = c_3/c_1$,

$\mathrm{Var}(M) = c_3(1 - c_2)/(c_1)^2$,

$\mathrm{E}(N) \qquad = c_2/c_1$,

$\mathrm{Var}(N) \qquad = c_2(1 - c_3)/(c_1)^2$,

$\mathrm{Cov}(M, N) = c_2 c_3/(c_1)^2$.

$\mathrm{E}(\mathrm{TNE}_r)$ and $\mathrm{E}(\mathrm{TLE}_r)$ are given by eq. 15. The variances of these statistics were derived by Chumbley (1972: appendix):

$$\mathrm{Var}(\mathrm{TNE}_r) = 2(2-a)/(3a)^2 + 1/(2a)^2$$

$$\mathrm{Var}(\mathrm{TLE}_r) = 4(76 + 24a - 27a^2)/[9a^2(2+a)(1+a)] + \mathrm{E}(\mathrm{TLE}_r) - \mathrm{E}^2(\mathrm{TLE}_r).$$

$\mathrm{E}(\mathrm{TNE}_v)$ and $\mathrm{E}(\mathrm{TLE}_v)$ were derived in the previous section and are given by eq. 18. $\mathrm{Var}(\mathrm{TNE}_v)$ and $\mathrm{Var}(\mathrm{TLE}_v)$ were similarly derived:

$$\mathrm{Var}(\mathrm{TNE}_v) = 2(100 + 96a + 31a^2 - 4a^3)/[(5+2a)^2(2+3a)^2]$$

$$\mathrm{Var}(\mathrm{TLE}_v) = 8(500 + 544a + 293a^2 + 51a^3 + 18a^4)/[9(5+2a)^2 \times (2+3a)^2].$$

As in the case of $\mathrm{E}(V)$ and $\mathrm{Var}(V)$, $\mathrm{E}(W)$ may be obtained by analyzing the following Markov chain, corresponding to the processing of a hypothesis based on one relevant and one irrelevant dimension until the occurrence of an infirming error:

$$\begin{array}{c} \\ \mathrm{A} \\ \mathrm{IR}_4 \\ \mathrm{IR}_{21} \\ \mathrm{IR}_{22} \\ \mathrm{IR}_1 \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathrm{A} & \mathrm{IR}_4 & \mathrm{IR}_{21} & \mathrm{IR}_{22} & \mathrm{IR}_1 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{a}{2} & \frac{3}{4}-\frac{a}{2} & 0 & 0 \\
\frac{1}{2} & \frac{a}{4} & 0 & \frac{1}{2}-\frac{a}{4} & 0 \\
\frac{1}{4} & 0 & \frac{a}{8} & \frac{a}{2} & \frac{3}{4}-\frac{5a}{8}
\end{bmatrix}
\end{array}
\begin{array}{c} \mathrm{P(C)} \\ \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{8} \\ \frac{1}{3} \end{bmatrix} \end{array} \tag{23}$$

$$\mathrm{I.V.} = [0 \quad 0 \quad 0 \quad 0 \quad 1].$$

From eq. 23 we get:

$$\mathrm{E}(\mathrm{TNE}_w) = \frac{2(16 + 61a + 61a^2 + 15a^3)}{3(2+a)(2+5a)(1+2a)},$$

$$\mathrm{Var}(\mathrm{TNE}_w) = \frac{2(230 + 1758a + 3157a^2 + 2044a^3 + 435a^4 - 100a^5)}{9(2+a)^2(2+5a)^2(1+2a)^2}$$

$$\mathrm{E}(\mathrm{TLE}_w) = 4(1+a)/(1+2a),$$

$$\mathrm{Var}(\mathrm{TLE}_w) = 4(3 + a + 2a^2)/(1+2a)^2.$$

Thus, predictions for the means and variances of TNE and TLE can be obtained by substitution of the above results in eqs. 21 and 22. The equations may then be used for parameter estimation purposes.

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